

Replication Code: A Simple Planning Problem for COVID-19 Lockdown ^{*}

Fernando Alvarez

University of Chicago and NBER

David Argente

Pennsylvania State University

Francesco Lippi

LUISS and Einaudi Institute for Economics and Finance

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Abstract

This short note contains documentation and matlab files that solve the planning problem using two different methods: interpolation and finite differences.

^{*}First draft, March 23, 2020. We thank Tom Phelan who shared his code of our problem using a similar version of the finite difference method, substantially improving the speed on our earlier method of computation.

1 Planning Problem

The replication files include matlab codes that solve the planning problem for COVID-19 lockdown studied in [Alvarez, Argente, and Lippi \(2020\)](#). We include files that use two different methods: interpolation and finite differences. We also include matlab files that, after loading a pre-existing output file, run a time-path simulation and constructs the figures in the paper for both methods. The next section provides more details on how we use the finite difference method to approximate the solution of the continuous time Bellman equation. It begins with a description of the finite difference method and describes how it can be implemented using the same grid size for the two state variables of the problem. It then describes its application when the grid size of the two state variables differs, which is how is implemented in the code for computational reasons. The files are provided for academic use. Users of the files (or modified version of it) should cite [Alvarez, Argente, and Lippi \(2020\)](#).

2 Finite Difference Method

The main idea of the finite difference method is to recast the problem of the planner as a stochastic control problem, where the domain is partitioned in space and in time, and whose solution is sure to converge to the continuous time version of the problem and the appropriate regularity conditions. We opted to numerically solve the value function of this problem for two reasons. First, the planning problem is not a convex problem. Second, we want to have a solution over the entire state space for counterfactuals.¹ We rely on discretization methods for stochastic control problems in continuous time described in [Kushner and Dupuis \(2013\)](#). In this note, we first describe how the finite difference method can be applied when the same grid size is used for both state variables of the problem in a triangular domain. We also use that the value function is known when either of the two state variables is zero (i.e. we

¹Alternatively, we could have used the shooting algorithm for the system of ordinary differential equations given by the state and co-states defined by the Hamiltonian, but that will provide us with conditions that are necessary but, in general, not sufficient.

solve the problem in the remaining part of the state space). We then describe how it can be implemented when the grid size is different. For computational reasons described in this note, this is our preferred methodology and the one that is implemented in the replication code. Since the original problem is deterministic, we use the “upwind” approximation method both in the interior and in the boundary of the state space, where we do not know the value function. We also discuss how to easily carry the minimization even when the first order conditions may not apply. We provide the details of the implementation in the rest of the note.

The planner solves the following Bellman-Hamilton-Jacobi equation:

$$\begin{aligned}
(r + \nu)V(S, I) = \min_{L \in [0, \bar{L}]} wL \left[\tau(S + I) + 1 - \tau \right] + I\phi(I) vsl + \\
+ [\beta S I (1 - \theta L)^2] [\partial_I V(S, I) - \partial_S V(S, I)] \\
- \gamma I \partial_I V(S, I)
\end{aligned} \tag{1}$$

The domain of V is $(S, I) \in \mathbb{R}^2$ such that $S + I \leq 1$. Note that $V(S, I)$ can be interpreted as the minimum expected discounted cost of following the optimal policy in units of output loss. Recall that if $\phi(I) = 0$, then the value of the objective of the planner at time $t = 0$ will give $N(0)w$. Thus, the quantity $rV(S, I)/w$ converts the stock-value of the value function into a ratio of the flow cost relative to output at time $t = 0$, when there is no virus. Finally, we notice that the value function has simple analytic expressions on the boundary of its domain, where the lockdown policy is not exercised. In particular, on the $I = 0$ axis we have $V(S, 0) = 0$, for all $S \in (0, 1)$, so that the cost is zero if nobody is infected. On the $S = 0$ axis we have $V(0, I) = vsl \left(\frac{\varphi\gamma}{r+\nu+\gamma} + \frac{\kappa\gamma I}{r+\nu+2\gamma} \right) I$ for all $I \in (0, 1)$.

Let S_i be the values on a grid of for S and I_j the values on a grid for I . Then we can try

at each interior grid point:

$$(r + \nu)V(S_i, I_j) = \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] + I_j \phi(I_j) vsl \quad (2)$$

$$+ [\beta S_i I_j (1 - \theta L)^2] [V_I^+(i, j) - V_S^-(i, j)] - \gamma I_j V_I^-(i, j)$$

where for the first term we use

$$V_S^-(i, j) = \frac{V(S_i, I_j) - V(S_{i-1}, I_j)}{S_i - S_{i-1}} \text{ and } V_I^+(i, j) = \frac{V(S_i, I_{j+1}) - V(S_i, I_j)}{I_{j+1} - I_j} \quad (3)$$

For instance, assuming that $S_i - S_{i-1} = I_{j+1} - I_j = \Delta$

$$V_I^+(i, j) - V_S^-(i, j) = \frac{1}{\Delta} [V(S_i, I_{j+1}) - V(S_i, I_j) - V(S_i, I_j) + V(S_{i-1}, I_j)] \quad (4)$$

$$= \frac{1}{\Delta} [V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \quad (5)$$

We can also use in the last term, since the drift is negative:

$$V_I^-(i, j) = \frac{1}{\Delta} [V(S_i, I_j) - V(S_i, I_{j-1})] \quad (6)$$

The rule used for the derivatives, whether to use forward or backward difference, is whether the drift of the corresponding term is positive or negative. This is the rule used in [Kushner and Dupuis \(2013\)](#), “Numerical Methods for Stochastic Control Problems in Continuous Time” chapter 5 for the explanation in the case of drift (b is the drift of the state in the book notation) and no diffusion or degenerate case ($a = 0$ in the book notation). For instance, see Example 2, for ”Uncontrolled Deterministic Case”, or see section 5.3 “The general Finite Difference Method”, The Diagonal Case, equations (3.4) and (3.5). The important point is that the direction of the change on the finite difference approximating the derivatives is given by the sign of the drift itself.

We will use the grids:

$$Grid_S = \{0, \Delta, 2\Delta, \dots, (N-1)\Delta\} \quad (7)$$

$$Grid_I = \{0, \Delta, 2\Delta, \dots, (N-1)\Delta\} \quad (8)$$

with $S_1 = 0, S_2 = \Delta, \dots, S_i = (i-1)\Delta, \dots, S_N = (N-1)\Delta = 1$, so $\Delta = \frac{1}{N-1}$. Likewise, $I_1 = 0, I_2 = \Delta, \dots, I_j = (j-1)\Delta, \dots, I_N = (N-1)\Delta = 1$. We require $S_i + I_j \leq 1$ so $(i+j-2)/(N-1) \leq 1$ or $(i+j-2) \leq N-1$ or $i+j \leq N+1$. We know the value function at $V(S_1, I_j)$ for all j given by the quadratic expression above $V(S_1, I_j) = vsl \left(\frac{\varphi\gamma}{r+\nu+\gamma} + \frac{\kappa\gamma I_j}{r+\nu+2\gamma} \right) I_j$. We also know that $V(S_i, I_1) = 0$ for all i . In both cases, i.e. when $I_1 = 0$ or when $S_1 = 0$, the optimal policy is no lockdown, i.e $L(1, j) = L(i, 1) = 0$ for all i, j , where we use $L(i, j)$ to denote the optimal lockdown policy when $S = S_i$ and $I = I_j$. Thus the expression for $V_I^-(i, j)$, and for $V_I^+(i, j) - V_S^-(i, j)$ are only computed for pairs (i, j) : $2 \leq i \leq N-1$ and $2 \leq j \leq N+1-i$.

Denote the right hand side of the HJB equation evaluated at (i, j) as $F(L; i, j)$ given by.

$$F(L; i, j) = Lw \left[\tau(S_i + I_j) + 1 - \tau \right] + I_j \phi(I_j) vsl \quad (9)$$

$$+ [\beta S_i I_j (1 - \theta L)^2] [V_I^+(i, j) - V_S^-(i, j)] - \gamma I_j V_I^-(i, j) \quad (10)$$

The first derivative is:

$$\frac{dF(L; i, j)}{dL} = w \left[\tau(S_i + I_j) + 1 - \tau \right] - \theta(1 - \theta L) 2 [\beta S_i I_j] [V_I^+(i, j) - V_S^-(i, j)] \quad (11)$$

and the second derivative is

$$\frac{d^2 F(L; i, j)}{dL^2} = \theta^2 2 [\beta S_i I_j] [V_I^+(i, j) - V_S^-(i, j)] \quad (12)$$

The following gives the optimal L :

1. If $V_I^+(i, j) - V_S^-(i, j) < 0$, then F is concave in L , and the minimum must be at either $L(i, j) = 0$ or $L(i, j) = \bar{L}$, and must be located by evaluating F at these two values.
2. If $V_I^+(i, j) - V_S^-(i, j) \geq 0$, then F is convex in L , and the minimum can be located using the first order condition $\frac{dF(L; i, j)}{dL} = 0$ and complementary slackness. Define

$$L^0(i, j) = \frac{-w \left[\tau(S_i + I_j) + 1 - \tau \right] + 2\theta [\beta S_i I_j] [V_I^+(i, j) - V_S^-(i, j)]}{2\theta^2 [\beta S_i I_j] [V_I^+(i, j) - V_S^-(i, j)]} \quad (13)$$

$$= -\frac{w \left[\tau(S_i + I_j) + 1 - \tau \right]}{2\theta^2 [\beta S_i I_j] [V_I^+(i, j) - V_S^-(i, j)]} + \frac{1}{\theta} \quad (14)$$

and

$$L(i, j) = \min \{ \bar{L}, \max \{ 0, L^0(i, j) \} \} \quad (15)$$

Note we can first multiply our discrete HJB equation by dt in both sides, and after add $V(S_i, I_j)$ to obtain:

$$\begin{aligned} [1 + (r + \nu)dt] V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\ &\quad [\beta S_i I_j (1 - \theta L)^2] \, dt [V_I^+(i, j) - V_S^-(i, j)] - \gamma I_j \, dt V_I^-(i, j) + V(S_i, I_j) \end{aligned}$$

Replacing the finite difference approximations:

$$\begin{aligned} [1 + (r + \nu)dt] V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\ &\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta} [V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \\ &\quad - \gamma I_j \frac{dt}{\Delta} [V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j) \end{aligned}$$

Moving some terms from the left to the right-and side:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta} [V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \\
&\quad - \gamma I_j \frac{dt}{\Delta} [V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j) [1 - (r + \nu)dt]
\end{aligned}$$

Collecting terms on the RHS

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [1 - (r + \nu)dt] \left\{ \frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} [V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \right. \\
&\quad \left. - \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} [V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j) \right\}
\end{aligned}$$

Further rearranging the terms:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [1 - (r + \nu)dt] \left[1 - 2 \frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_i) \\
&\quad + [1 - (r + \nu)dt] \left[\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] [V(S_i, I_{j+1}) + V(S_{i-1}, I_j)] \\
&\quad + [1 - (r + \nu)dt] \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} V(S_i, I_{j-1})
\end{aligned}$$

which can be rearrange to obtain the following expression of interest:

$$\begin{aligned}
V(S_i, I_j) = & \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
& + [1 - (r + \nu)dt] \left[1 - 2 \frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_i) \\
& + [1 - (r + \nu)dt] \left[\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j+1}) \\
& + [1 - (r + \nu)dt] \left[\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_{i-1}, I_j) \\
& + [1 - (r + \nu)dt] \left[\frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j-1})
\end{aligned}$$

When i, j are such that $S_i + I_j = 1$, so that the state is in the boundary, or equivalently that $i + j = N + 1$, we can't evaluate $V(S_i, I_{j+1})$, so we use a different scheme for the difference. We approximate the difference in the derivatives as follows: $\partial_I V(S_i, I_j) - \partial_S V(S_i, I_j) = (V(S_{i-1}, I_{j+1}) - V(S_i, I_j)) / \Delta$. The resulting operator is:

$$\begin{aligned}
V(S_i, I_j) = & \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
& + [1 - (r + \nu)dt] \left[1 - \frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_i) \\
& + [1 - (r + \nu)dt] \left[\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_{i-1}, I_{j+1}) \\
& + [1 - (r + \nu)dt] \left[\frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j-1})
\end{aligned}$$

Note that the expressions for $V(S_i, I_j)$ can be considered as the Bellman equation corresponding to a finite state stochastic discrete time programming problem. Indeed, we can use it to define a corresponding operator and use the method of successive approximations to approximate its fixed point. The optimal value of $L(i, j)$ was obtained above, as function of the difference of the finite difference approximation on derivatives. On the one hand, note that for any fixed grid side $\Delta > 0$, the last terms of the left hand side can be considered a weighted average of the value function at different points. For dt small enough all the weights

are non-negative. On the other hand, this operator contracts at rate $[1 - (r + \nu)dt]$, which is closer to one as dt gets smaller.

To make sure that $\left[1 - 2\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta}\right] > 0$ note that

$$\begin{aligned} 2\frac{\beta S_i I_j (1 - \theta L)^2}{1 - (r + \nu)dt} \frac{dt}{\Delta} &\leq \frac{\beta dt (N - 1)}{2[1 - (r + \nu)dt]} \\ \frac{\gamma I_j}{1 - (r + \nu)dt} \frac{dt}{\Delta} &\leq \frac{\gamma dt (N - 1)}{1 - (r + \nu)dt} \end{aligned}$$

Thus we require dt to satisfy:

$$\begin{aligned} 1 &\geq [r + \nu + (N - 1)(\beta/2 + \gamma)] dt \text{ or} \\ dt &\leq \frac{1}{r + \nu + (N - 1)(\beta/2 + \gamma)} \end{aligned}$$

Different S and I grid sizes. Since for the initial conditions of interest, the state spends a large time around low values of I . We consider a non-equal spaced grid for I , while keeping an equally spaced grid for S such that:

$$\begin{aligned} S_{i+1} &= S_i + \Delta_S = i\Delta_S \text{ for } i = 1, 2, \dots, N_I \\ I_{j+1} &= I_j + \Delta_I = j\Delta_I \text{ for } j = 1, 2, \dots, N_S \\ \Delta_S &= k \Delta_I \text{ where } k \text{ is a strictly positive integer} \end{aligned}$$

Thus, we have $\Delta_S = \frac{1}{N_S - 1}$ and $\Delta_I = \frac{1}{N_I - 1}$ and $(N_S - 1)k = N_I - 1$ or $N_S = (N_I - 1)/k + 1$.

We will assume that $N_S, \gamma/\beta$ satisfies:

$$\Delta_S = \frac{1}{N_S - 1} \leq \frac{\gamma}{\beta}$$

We will consider two cases:

1. In the “interior” of the state space we will estimate the derivatives by:

$$V_S^-(i, j) = \frac{1}{\Delta_S} [V(S_i, I_j) - V(S_{i-1}, I_j)] \quad (16)$$

$$V_I^+(i, j) = \frac{1}{\Delta_I} [V(S_i, I_{j+1}) - V(S_i, I_j)] \quad (17)$$

$$V_I^-(i, j) = \frac{1}{\Delta_I} [V(S_i, I_j) - V(S_i, I_{j-1})] \quad (18)$$

Replacing them in the value function, for values $S_i > 0$ and $I_{j-1} > 0$ are in the state space, but also so that $S_i + I_{j+1} \leq 1$. This corresponds to:

$$2 \leq i \leq N_S - 1$$

$$2 \leq j \leq N_I - 1 - (i - 1)k$$

since $S_i + I_{j+1} \leq 1$ is equivalent to $j\Delta_I + (i - 1)\Delta_S \leq 1$ or $j \leq \frac{1}{\Delta_I} - (i - 1)\frac{\Delta_S}{\Delta_I}$ or $j \leq N_I - 1 - (i - 1)k$.

Replacing the finite difference approximation into the value function:

$$\begin{aligned} (r + \nu)V(S_i, I_j) = & \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] + I_j \phi(I_j) vsl \\ & + [\beta S_i I_j (1 - \theta L)^2] \frac{1}{\Delta_I} [V(S_i, I_{j+1}) - V(S_i, I_j)] \\ & - [\beta S_i I_j (1 - \theta L)^2] \frac{1}{\Delta_S} [V(S_i, I_j) - V(S_{i-1}, I_j)] \\ & - \gamma I_j \frac{1}{\Delta_I} [V(S_i, I_j) - V(S_i, I_{j-1})] \end{aligned}$$

Multiplying by dt and adding $V(S_i, I_j)$ on each side:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_I} [V(S_i, I_{j+1}) - V(S_i, I_j)] \\
&\quad - [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} [V(S_i, I_j) - V(S_{i-1}, I_j)] \\
&\quad - \gamma I_j \frac{dt}{\Delta_I} [V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j) [1 - (r + \nu)dt]
\end{aligned}$$

Grouping common terms for the value functions evaluated at the same points:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + \left\{ [1 - (r + \nu)dt] - [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_I} - [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} - \gamma I_j \frac{dt}{\Delta_I} \right\} V(S_i, I_j) \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_I} V(S_i, I_{j+1}) \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} V(S_{i-1}, I_j) \\
&\quad + \gamma I_j \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

taking $[1 - (r + \nu)dt]$ as a common factor:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [1 - (r + \nu)dt] \left\{ 1 - \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} - \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_S} - \frac{\gamma I_j}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} \right\} V(S_i, I_j) \\
&\quad + [1 - (r + \nu)dt] \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} V(S_i, I_{j+1}) \\
&\quad + [1 - (r + \nu)dt] \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_S} V(S_{i-1}, I_j) \\
&\quad + [1 - (r + \nu)dt] \frac{\gamma I_j}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

2. Now we write the approximation for those (i, j) for which $S_i + I_j = 1$ so that $S_i + I_{j+1} >$

1. We will first consider the case where

$$2 \leq i \leq N_S - 1 \text{ and } j = N_I - (i - 1)k$$

For this case we use

$$\begin{aligned} V(S - \Delta_S, I + k\Delta_I) - V(S, I) &= -\partial_S V(S, I)\Delta_S + \partial_I V(S, I)\Delta_I \\ &= \Delta_S [\partial_I V(S, I) - \partial_S V(S, I)] + o(\|(\Delta, \Delta_I)\|) \end{aligned}$$

Thus we approximate $\partial_I V(S, I) - \partial_S V(S, I)$ as

$$V_I^+(i, j) - V_S^-(i, j) \equiv \frac{1}{\Delta_S} [V(S_{i-1}, I_{j+k}) - V(S_i, I_j)]$$

where k is the ratio of Δ_S/Δ_I so $S_{j-1} + I_{j+k} \leq 1$ whenever $S_i + I_j = 1$. For $\partial_I V(S, I)$ we still use:

$$V_I^-(i, j) = \frac{1}{\Delta_I} [V(S_i, I_{j-1}) - V(S_i, I_j)]$$

Replacing the finite difference into the value function:

$$\begin{aligned} (r + \nu)V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] + I_j \phi(I_j) vsl \\ &\quad + [\beta S_i I_j (1 - \theta L)^2] [V_I^+(i, j) - V_S^-(i, j)] - \gamma I_j V_I^-(i, j) \end{aligned}$$

Using the expressions for the finite differences, multiplying by dt and adding $V(S_i, I_j)$

on each side:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} [V(S_{i-1}, I_{j+k}) - V(S_i, I_j)] \\
&\quad - \gamma I_j \frac{dt}{\Delta_I} [V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j) [1 - (r + \nu)dt]
\end{aligned}$$

Grouping common terms for the value functions evaluated at the same points:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + \left\{ [1 - (r + \nu)dt] - [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} - \gamma I_j \frac{dt}{\Delta_I} \right\} V(S_i, I_j) \\
&\quad + [\beta S_i I_j (1 - \theta L)^2] \frac{dt}{\Delta_S} V(S_{i-1}, I_{j+k}) \\
&\quad + \gamma I_j \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

taking $[1 - (r + \nu)dt]$ as a common factor:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L \in [0, \bar{L}]} Lw \left[\tau(S_i + I_j) + 1 - \tau \right] dt + I_j \phi(I_j) vsl \, dt \\
&\quad + [1 - (r + \nu)dt] \left\{ 1 - \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_S} - \frac{\gamma I_j}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} \right\} V(S_i, I_j) \\
&\quad + [1 - (r + \nu)dt] \frac{[\beta S_i I_j (1 - \theta L)^2]}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_S} V(S_{i-1}, I_{j+k}) \\
&\quad + [1 - (r + \nu)dt] \frac{\gamma I_j}{[1 - (r + \nu)dt]} \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

References

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